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of the XVth Century

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Extrait d'*ISIS*, n° 11 (vol. IV, 2) October 1921.

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Am. Arbo.
1920

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A dissertation submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in the University of Michigan,
issued in four parts in *Isis*, vol. IV, pp. 295-323, vol. IV, pp. 459-465, vol.
V, pp. 99-115, and vol. V, pp. 339-363.

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The development of trigonometric methods down to the close of the XVth century.

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This paper serves as an introduction to a study on RICHARD WALLINGFORD'S *Quadripartitum* which will be published (with notes and translation) in an ulterior number of *Isis*. My study of the *Quadripartitum* was undertaken in the summer of 1918 at the suggestion of Professor L. C. KARPINSKI of the University of Michigan, who had himself made a preliminary examination of the manuscript. The introduction is largely based on the late A. VON BRAUNMÜHL'S *Vorlesungen*. It treats the evolution of the methods of plane trigonometry from the earliest times to the close of the fifteenth century. However, since WALLINGFORD makes the methods of calculating tables of sines and chords the central feature of Part I of the *Quadripartitum*, it has seemed advisable not to limit in point of time the treatment of the methods of constructing tables of natural sines.

The whole work, as published in this and following numbers of *Isis*, was submitted as a thesis in partial fulfillment of the requirements for the degree of Doctor in Philosophy in the University of Michigan, 1920. To Professor L. C. KARPINSKI my thanks are due for very helpful directions and criticism throughout the preparation of this dissertation.

In the construction of their pyramids, nearly all of which had their faces inclined approximately 52° to the horizontal, the ancient Egyptians employed the practical equivalent of trigonometric functions. As interpreted by CANTOR (1). $DF = \text{pir em us}$ (fig. 1),

(1) CANTOR, *Vorlesungen über Geschichte der Mathematik*, 2nd Edition, vol. 1, p. 59-60. Leipzig, 1898,

$DC = qai\ en\ haru$, $2CF = uxa\ tebt$, $2CE = senti$. But *seqt*, « the ratio number », in one plane is $\frac{1/2\ uxa\ tebt}{piremus} = \frac{CF}{DF} = \cos DFC$ and in the other it is $\frac{qai\ en\ haru}{1/2\ senti} = \frac{CD}{CE} = \tan DEC$. The stone-cutters of the pyramids probably used the term *seqt* somewhat as modern carpenters use the word *pitch* in sawing rafters. « The sockets cut into the limestone surface of the desert plateau in which the cornerstones of the Great Pyramid were laid... enabled PETRIE to establish the length of the sides as 755 feet. The maximum error he found to be 0.63 of an inch, that is, less than one fourteen-thousandth of the total length of the side. The error of angle at the corners he found to be 12" of a degree, that is, one twenty-seven-thousandth of the right angle which the architect had laid out at the corner » (1).

With this glimpse at the earliest trace of our science we must turn to astronomy or astrology for further development. In fact it was only with the object of perfecting astronomical calculations that trigonometry was cultivated at all for some 3000 years. The first independent treatise on trigonometry is that by AT-TUSI of Bagdad in the thirteenth century, while two centuries more passed before a similar work appeared in Europe. Because of the interest in astronomy, spherical trigonometry was developed prior to the plane, but we shall limit our discussion to the latter.

A. VON BRAUNMÜHL (2) remarks that while in a papyrus of the early Egyptians was discovered the first trigonometric ratio we possess only fragmentary records of the early astronomical observations and computations. Doubtless the Egyptians and, probably earlier, the Chaldeans cultivated with devotion the pseudo-science astrology, as related by THEON OF SMYRNA. Among the early Chinese there are also traces of some idea of trigonometric ratio. But the first noteworthy development is to be found among the Greeks, who built their scientific structure upon the foundations erected by the Egyptians and the Babylonians. Indeed the earliest Greek mathematicians and astronomers were travellers who acquired their knowledge at the feet of

(1) BREASTED, « The Origins of Civilization », *Scientific Monthly*, p. 91-92 January, 1920,

(2) A. VON BRAUNMÜHL, *Vorlesungen über Geschichte der Trigonometrie*, vol. 1, p. 3-6. Leipzig, 1900. Later references to this volume will be devoted simply by BRAUNMÜHL.

Egyptian and Babylonian priests. Thus THALES (640-548 B. C.) of Miletus, who spent many years in Egypt and became the first prominent mathematician among the Greeks, learned to determine from its shadow the height of a pyramid and from the similarity of triangles the distance of a ship from the harbor of Miletus.

ANAXIMANDER, contemporary and pupil of THALES, appears to have introduced into Greece the *gnomon* (sun-dial) of the Babylonians : from the lengths of the gnomon and its shadow the determination of the height of the sun leads to the tangent of its altitude.

But it is in the little book of 18 propositions *On the Sizes and Distances of the Sun and Moon* by ARISTARCHUS (about 310-230 B. C.) of Samos that we first find an attempt to calculate trigonometric ratios. HEATH (1) says : « Besides the formal assumptions at the beginning of the treatise, there lie at the root of ARISTARCHUS' reasoning certain propositions assumed without proof, presumably because they were generally known to the mathematicians of the day. The most general of the propositions are equivalent to the statements — If α is what we call the circular measure of an angle, and α is less than $\pi/2$, then (I) the ratio $\sin \alpha/\alpha$ decreases as α increases from 0 to $\pi/2$, but (II) the ratio of $\tan \alpha/\alpha$ increases as α increases from 0 to $\pi/2$ ». In propositions 7, 11, 12, 13, respectively, ARISTARCHUS reaches conclusions equivalent to the following : $1/18 > \sin 3^\circ > 1/20$, $1/45 > \sin 1^\circ > 1/60$, $1 > \cos 1^\circ > 89/90$, $1 > \cos^2 1^\circ > 44/45$ (2).

The statement of proposition 7, for example, is : « The distance of the sun from the earth is greater than eighteen times, but less than twenty times, the distance of the moon from the earth ». The assumption here is that at dichotomy the angle at the sun is 3° .

HIPPARCHUS, called by the Greeks the Father of Astronomy, was born at Nicea and lived in Rhodes and Alexandria during the period 161 to 126 B. C. That he was also the founder of the trigonometry of chords appears from two sources. THEON (about 365 A. D.) of Alexandria says that HIPPARCHUS wrote a treatise in 12 books on the calculation of chords in a circle, but no trace of this book has been found. Furthermore one work of HIPPARCHUS from a commentary on the astronomical work of EUDOXUS and ARATUS has come down to us, in which he says that he had proved graphically (PETAVIUS' translation

(1) HEATH, *Aristarchus of Samos*, p. 333, Oxford. 1913.

(2) BRAUNMÜHL, incorrectly gives here $1 > \cos 1^\circ > 44/45$, p. 8.

is « per lineas ») in an earlier work all that he used in the commentary (1).

From THEON we also learn that MENELAUS (about 98 A. D.) of Alexandria treated in 6 books the trigonometry of chords, but this book is also lost. However, Arabic and Hebrew translations of the *Spherics* of MENELAUS in 3 books have come down to us (2). Similar works of ANAXIMANDER (about 611 B. C.), AUTOLYCUS (about 330 B. C.), EUCLID (about 320 B. C.), HYPsicLES (about 180 B. C.) of Alexandria, and THEODOSIUS (first century B. C.), of Tripoli contain not a trace of trigonometric computation. But MENELAUS' *Spherics* gives in great completeness the geometric theorems on which Greek trigonometry is built and in the last book treats of the relations between chords and arcs. The first theorem in the second book, generally known as the Theorem of MENELAUS, is the foundation of the laws of the spherical trigonometry that obtained until trigonometry twelve or fourteen centuries later broke the shackles that bound it to astronomy.

The Theorem of MENELAUS may be stated thus : The product of the three ratios of the consecutive segments of the sides of a plane triangle made by any rectilinear transversal equals unity.

Let the transversal intersect the sides of triangle ABC in the points D, E, F (fig. 2). Dropping from A, B, C perpendiculars AM, BN, CP to line FD, we have by similar triangles $\frac{AD}{DB} = \frac{AM}{BN}$, $\frac{BE}{EC} = \frac{BN}{CP}$, $\frac{CF}{FA} = \frac{CP}{AM}$.

Whence by multiplication $\frac{AD}{DB} \cdot \frac{BE}{EC} \cdot \frac{CF}{FA} = 1$, or as it is often written, $AD \cdot BE \cdot CF = DB \cdot EC \cdot FA$.

Now it is to be observed that if we consider the figure ABCDEF a complete quadrilateral, there are involved four distinct triangles and a transversal corresponding to each. But only two of these triangles are considered by MENELAUS, PTOLEMY, and their followers, for as we shall see in the next paragraph, their point of view was wholly different from ours. By them one ratio was always equated to the product of the other two. For example, $\frac{AD}{DB} = \frac{EC}{BE} \cdot \frac{CF}{FA}$, which is always stated

(1) See BRAUNMÜHL, « Beiträge zur Geschichte der Trigonometrie », *Abh. d. Kais. L.-P. Deutschen Akad. d. Naturforscher*, vol. LXXI. n. 1, p. 4, Halle, 1897.

(2) BJÖRNBO, « Studien über MENELAOS' Sphärik », *Abhandlungen zur Geschichte d. Math. Wis.*, vol. 14, p. 10. Leipzig, 1902.

in words just as we find it in the *Quadripartitum*: « The ratio of AD to DB is composed of the ratio of CE to BE and of the ratio of CF to FA » (1). If in the equation $AD \cdot BE \cdot CF = DB \cdot EC \cdot FA$ we divide by the product of BE, CF and each of the factors on the right successively, we obtain 3 distinct « composed ratios ». By interchanging the denominators on the right of these we obtain 3 other forms. Then because 6 different forms occur with each of the antecedents AD, BE, CF there are in all 18 « modes » to use the classic term employed by WALLINGFORD. THABIT IBN QURRAH (836-901, Bagdad) wrote a work on the « 18 modes » which is extant in the Arabic text and in Hebrew translations, besides Latin versions (2). In his *Liber Abaci* (1202) LEONARD OF PISA translated the Arabic *al-katta* (the cutter) as *figura cata* (3). Later writers evidently adopted simply the term *cata*, for we often find WALLINGFORD speaking of « arranging the *catha* » (or *chata* or *chatha*) for the solution of a particular problem. The ancient theory of the quadrilateral reaches its highest development in AT-TUSI's *Treatise on the Quadrilateral* (4).

The medieval point of view is found in the *Quadripartitum* of which Part III is devoted entirely to the treatment of the quadrilateral. WALLINGFORD says: « I shall draw the two lines AC and AD (fig. 3) from the point A and two other lines CE and DB from the extremities C and D and intersecting in a point G. Then I say that the ratio of AC to AB is composed of the ratio of CE to GE and of the ratio of DG to DB. From B I shall draw a line BF parallel to CG and in the proportion put GE between CE and BF. Then by the argument of Part II 2 (5) the ratio of CE to BF is composed of the ratios CE to GE

(1) The *regula sex quantilatum*, as this theorem was known when applied to the spherical quadrilateral, is obtained by changing the above lines to arcs and writing « The ratio of chord (2AD) to chord (2DB) is composed of the ratio of chord (2CE) to chord (2BE) and of the ratio of chord (2CF) to chord (2FA) ». It is well to observe that the period covered in this dissertation antedates the symbolic notation and the equation of modern algebra.

(2) BRAUNMÜHL, p. 46.

(3) CANTOR, *Vorlesungen*, vol. 2, p. 15, note 3.

(4) *Traité du Quadrilatère* attribué à NASSIRUDDIN-EL-TOUSSY d'après un manuscrit tiré de la bibliothèque de S. A. CARATHEODORY, Constantinople, 1891. Copy used was loaned by Prof. ALEXANDER ZIWET of the University of Michigan.

(5) Part III is devoted to the application to the *catha* of the theory of the « six quantities » developed in Part II. To us today much of this seems quaint and laborious, for modern methods have greatly simplified the processes.

and GE to BF. But by EUCLID, VI 4 the ratio of GE to BF equals the ratio of DG to DB. Hence the ratio of CE to BF is composed of the ratios CE to GE and DG to DB. But it is known by VI 4, that the ratio of CE to BF equals the ratio of AC to AB. Therefore the ratio of AC to AB is composed of the ratios CE to GE and DG to DB ».

This is the first of 18 cases of *conjunctive catha*, corresponding to external section of the sides of the triangle; these are followed by the 18 cases of *disjunctive catha*, corresponding to internal section of two of the sides. In the second of his two figures PTOLEMY (1) says that

one shows « by division » (κατὰ διαίρεσιν) that $\frac{GE}{EA} = \frac{GZ}{DZ} \cdot \frac{DB}{BA}$ (fig. 4).

Evidently this is the source of the term *disjunctive* employed by WALLINGFORD. Thence naturally arose the opposite term for the former type.

We may pause here a moment to consider the purpose of the *Quadripartitum*. A perusal of Part I will show that the object in view is the calculation of tables of sines as given by AZ-ZARQALI and, incidentally, tables of chords as given by PTOLEMY. Similarly the ultimate purpose of the remainder of the work is evidently the making of astronomical calculations with the use of the tables of sines. But although the author's primary interest in the work was astronomical, the form of it and various remarks show clearly that he was imbued with the scientific spirit and that he keenly felt the need of a solid mathematical foundation for such work. He complains of the lax methods of his predecessors. For example, in connection with his proof of the Theorem of MENELAUS for the spherical quadrilateral (p. 27, line 11) we read: « Then because the figure of the demonstration varies in three ways and PTOLEMY touched only one of them, in order that we may leave nothing incomplete... let us pursue all three modes » (2). Again (p. 33, line 8) he says: « But first observe that in the *Canones* of ARZACHEL (AZ-ZARQALI) and others the doctrine of this operation (of finding celestial latitude) is not to multiply mean by mean and divide by the first (term of the proportion) but to multiply the first by the third and to divide by the second... » Clearly WALLINGFORD is here endeavoring to make sure of the mathematical basis of his own work.

(1) *Almagest*, Edition HALMA, vol. 1, p. 51.

(2) Compare BJÖRNBO, « Studien über MENELAOS' Sphärik », *Abhandlungen zur Gesch. d. Math. Wis.*, vol. 14, p. 88-92. Leipzig, 1902.

CLAUDIUS PTOLEMY who made astronomical observations in Alexandria during the period 125-161 composed in Greek a work of 13 books commonly called the *Almagest* (1) which holds in astronomy the place held in pure mathematics by the *Elements* of EUCLID. It is safe to say that all astronomical tables for more than a millennium were based on the *Almagest*. We are chiefly concerned with the first book, in which the mathematical foundation of the work is laid.

PTOLEMY divides the circumference into 360 *degrees* and the diameter into 120 *parts* and then subdivides each *part* into *minutes*, *seconds*, *thirds*, etc. (*partes minutæ primæ*, *partes minutæ secundæ*, *partes minutæ tertiæ*, etc.) in the sexagesimal system of the Babylonians with which we are today familiar in the subdivisions of the degree and the hour. He assumes the ratio of the circumference of a circle to its diameter to be $3^{\circ} 8' 30''$, that is $\pi = 3 + 8/60 + 30/60^2 = 377/120 = 3.14166$ (2).

In the *Almagest* are found 4 fundamental principles used by PTOLEMY to calculate his table of chords for intervals of $30'$ of arc (3).

(I) By the theorems of EUCLID are found the sides of the regular triangle, quadrilateral, pentagon, hexagon, and decagon in *parts* of which 60 make the radius. In reverse order the values obtained are : $\text{chd } 36^{\circ} = 37^{\text{p}} 4' 55''$, $\text{chd } 60^{\circ} = 60^{\text{p}}$, $\text{chd } 72^{\circ} = 70^{\text{p}} 32' 3''$, $\text{chd } 90^{\circ} = 84^{\text{p}} 51' 40''$, $\text{chd } 120^{\circ} = 103^{\text{p}} 35' 23''$. Since the chords of supplementary arcs form with the diameter a right triangle, $\text{chd } 144^{\circ} = \sqrt{120^2 - (\text{chd } 36^{\circ})^2}$, and so for the supplement of any arc whose chord is known.

(II) The so-called Theorem of PTOLEMY, « The rectangle (product) of the diagonals of an inscribed quadrilateral equals the sum of the rectangles of the opposite sides », given in the form of a lemma, is used to find the chord of the sum or difference of two arcs. This theorem furnishes the results that we obtain by the addition and subtraction formulas. After proving the general theorem PTOLEMY uses the special case when one side is the diameter to find the chord of the difference of two arcs. To show how closely this is related to our trigonometry, we have (fig. 5) : $BG, AD = AG, BD - AB, GD$, that is $\text{chd } (AG - AB)$, $2r = \text{chd } AG$, $\text{chd } (180^{\circ} - AB) - \text{chd } AB$,

(1) All our references are to HALMAS edition, Paris, 1813, which contains the Greek text and a French translation.

(2) *Almagest*, p. 413.

(3) *Almagest*, p. 28-36. See also BRAUNMÜHL, p. 19-21.

chd $(180^\circ - \text{AG})$, which is PTOLEMY's formula. If now we put angle $\text{AEG} = 2\alpha$, angle $\text{AEB} = 2\beta$, and $r = 1$, we have $\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$ (1). The fundamental difference is that our notion of ratio does not enter into the trigonometry of the Greeks or the Arabs. They dealt with the lengths of chords or half-chords (sinus) in a circle of known radius and it was more than eight centuries after PTOLEMY that the thought of making the radius unity occurred to an Arab, ABU'L-WEFA; still later by about six centuries a European, BÜRGI, conceived a like notion (2).

Similarly PTOLEMY finds from the chords of two given arcs the chord of their sum. Thus given (fig. 6) arcs AB, BG with known chords a, b , to find x , the chord of their sum. We have $x = \text{AG} = \sqrt{4r^2 - \text{GD}^2}$ and $\text{GD} = \frac{(a' \cdot b' - a \cdot b)}{2r}$. If we put $\text{AZB} = 2\alpha$, $\text{BZG} = 2\beta$, $r = 1$, the latter reduces to $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$.

(III) The chord of the half arc is found from that of the arc as follows (fig. 7): Let $\text{DB} = a$ be the given chord, G the mid-point of arc BD, and AD a diameter. Make $\text{AE} = \text{AB} = a'$ and draw GZ perpendicular to AD. Then $\text{EG} = \text{BG} = \text{GD}$ and $\text{EZ} = \text{ZD} = \frac{(2r - a')}{2}$ where $a = \sqrt{4r^2 - a'^2}$. Whence PTOLEMY gets finally $\text{GD} = \sqrt{2r \cdot \text{ZD}}$. Making $r = 1$, arc $\text{BD} = 2\alpha$, and from the value of ZD obtaining $\text{GD} = \sqrt{r(2r - a')}$, we have $\sin \frac{\alpha}{2} = \sqrt{\left(\frac{1 - \cos \alpha}{2}\right)}$. By the use of this theorem are found from the chord of 12° the values $\text{chd } 3/2^\circ = 1^\circ 34' 15''$, $\text{chd } 3/4^\circ = 0^\circ 47' 8''$.

(IV) By means of an ingenious method of interpolation based on the principle (1) used by ARISTARCHUS, PTOLEMY finds $\text{chd } 1^\circ = 1^\circ 2' 50''$. Then by III. $\text{chd } 1/2^\circ = 0^\circ 31' 25''$. He now has the complete means of calculating his table of chords for differences of $30'$ of arc, with which he contents himself. We find that his table of chords is accurate to at least five decimal places throughout. For instance set-

(1) Compare TROPFKE, *Geschichte der Elementer Mathematik*, vol. 2, p. 226. Leipzig, 1903.

(2) For the development of the idea of ratio compare JACKSON, *Sixteenth Century Arithmetic in Columbia Contributions to Education*, Teachers College, Series 8.

ting $r = 1$ and observing that $1^p = 1/60$, we have $\sin 30' = 1/2$, $\text{chd } 1^\circ = 0^p 31' 25'' = 1885/3600^p = 1885/216000 = 0.0087268$ to seven decimal places, which agrees to six places with the correct value 0.0087265.

Improvement of mathematical tables has naturally gone hand in hand with the development of the science, but for our purpose it seems fitting to unify our discussion of trigonometric tables. Hence we shall conclude the treatment of this phase of the subject before proceeding further with the general development of trigonometric methods.

In the *Surya-Siddhanta* the Hindus, assuming $\alpha = 225' = \sin \alpha$ and proceeding for intervals of $225' = 3^\circ 45'$, gave rules equivalent to the formula (1), $\sin(n+1)\alpha = \sin(n\alpha) + \sin \alpha - \frac{\sin \alpha}{\sin \alpha} - \frac{\sin 2\alpha}{\sin \alpha} - \frac{\sin 3\alpha}{\sin \alpha} - \dots - \frac{\sin(n\alpha)}{\sin \alpha}$. By subtraction this becomes

$$\sin(n+1)\alpha = \sin(n\alpha) + \left[\sin(n\alpha) - \sin(n-1)\alpha - \frac{\sin(n\alpha)}{\sin \alpha} \right],$$

where $\sin(n\alpha) - \sin(n-1)\alpha$ is the first difference and $\frac{\sin(n\alpha)}{\sin \alpha}$ is the second. Here $r = 3438'$ and the results are in error less than $1''$ in the accompanying table. That the table was not constructed from the rules, but conversely, is shown by the fact that the tabular values exceed by $1'$ those given in some instances by the rules.

Using the Hindu *sinus* or half-chord, the Arabs were able to improve the Ptolemaic tables. ABU'L-WEFA (940-998, Bagdad) calculated a table of sines for intervals of $15'$ in which the error for $\sin 30'$ is only one unit in the ninth decimal place (2). From the sines of 36° and 60° , by the bisection of angles, he readily obtains the values of $\sin 18/32^\circ$, $\sin 15/32^\circ$, and $\sin 12/32^\circ = \sin(36^\circ - 30^\circ)/16$ and then, using the fact that for $90^\circ > \alpha + \beta > \alpha > \beta > 0$, $\sin(\alpha + \beta) - \sin \alpha < \sin \alpha - \sin(\alpha - \beta)$, finds $\sin 16/32^\circ > \sin 15/32^\circ + 1/3 (\sin 18/32^\circ - \sin 15/32^\circ)$ and $\sin 16/32^\circ < \sin 15/32^\circ + 1/3 (\sin 15/32^\circ - \sin 12/32^\circ)$. The arithmetic mean gives $\sin 1/2^\circ = \sin 15/32^\circ + 1/6 (\sin 18/32^\circ - \sin 12/32^\circ)$,

(1) BURGESS, "Translation of the *Surya-Siddhanta*", *Journal of the American Oriental Society*, vol. 6, 1860, p. 196, ch. II, stanzas 15-16.

(2) WOEPCKE, "Mathématiques chez les Orientaux", *Journal Asiatique* Series V, vol. 15, 1860, p. 285 and fol.

a value too large by 0.00000 00010 00529 (1). The error arising from the formula of interpolation is about 6 times that of the quantities entering it but the error is only about $1/4$ of that of the formula derived from PTOLÉMY's method, namely, $\sin 1/2^\circ = 1/3(2 \sin 3/8^\circ + \sin 3/4^\circ)$ (2).

The last and probably the greatest of the Arabic astronomers was ULUG BEG (1393-1449, Samarcand) who found $\sin 1^\circ$ by means of his celebrated third-degree equation

$$x = \frac{\left(\frac{r^2}{4} \sin 3^\circ + x^3\right)}{\frac{3r^2}{4}}$$

in which he replaces x by $\sin 1^\circ$ and solves directly by a very ingenious method of approximation. For $r = 60$ the result in the sexagesimal system is of the form $x = a_0 + a_1/60 + a_2/60^2 + a_3/60^3 + \dots$, or as given in the *Prolégomènes*, $\sin 1^\circ = 1^\circ 2' 49'' 43''' 11^{iv}$, a result correct to within $1/4$ of 1^{iv} , that is within 4 units in the tenth decimal place (3). ULUG BEG's table of sines is constructed for intervals of one minute.

But in Europe little progress was made in advance of the tables of PTOLÉMY until REGIOMONTANUS, following PEURBACH's construction of a table of sines with radius 600.000, used the radius 6.000.000 and later made the great step forward of employing a radius of 10.000.000, thus paving the way for decimal values with unit radius. Using $\sin 3/4^\circ + 1/3 \sin 3/4^\circ > \sin 1^\circ > \sin 3/4^\circ + 1/6 (\sin 6/4^\circ - \sin 3/4^\circ)$

(1) For complete geometric proof see BRAUNMÜHL, p. 56-57. But observe typographical errors. In the second of the above inequalities $15/32^\circ$ appears as $18/32^\circ$. Correctly written, $46^v 40^{vi} 50^{vii} = 0.00000 00010 00529$ and $\sin 1/2^\circ = 0.00872 65364 98903$. TROPFKE (vol. 2, p. 298) and CAJORI (*History of Mathematics*, 1919, p. 106) both give this value of $\sin 1/2^\circ$ as correct to the ninth decimal place instead of the eighth.

(2) WOEPCKE, "Discussion de deux méthodes arabes pour déterminer une valeur approchée de $\sin 1^\circ$ ", *Journal de Mathématiques*, vol. 19, 1854, p. 159 and fol.

(3) SÉDILLOT, *Prolégomènes des Tables astronomiques d'OLUG-BEG*, p. 81 and fol. Paris, 1853. Full treatment of the theory involved is given in the above article by WOEPCKE, p. 167 and fol. See also BRAUNMÜHL, p. 73, 74. For $r = 1$ and $x = \sin 1^\circ$ the equation becomes the well known relation $\sin 3^\circ = 3 \sin 1^\circ - 4 \sin^3 1^\circ$. We have here an illustration of the dependence of the trisection problem upon the solution of an equation of the third degree.

and $r = 6,000,000$, by the formula for the half angle he gets $52360 > \sin 1/2^\circ > 52358$. Whence $\sin 1/2^\circ = 52359$. For $r = 1$ this becomes $\sin 1/2^\circ = 0.0087265$, correct to 7 decimal places. The table is constructed for intervals of one minute (1).

The *Opus Palatinum de Triangulis* of GEORG JOACHIM RHAETICUS (1514-1576), finished at Neustadt in 1596 by his pupil, VALENTIN OTHO (1550?-1605), contains the natural values to 10 places (*i. e.* $r = 10^{10}$) of the six trigonometric ratios for every $10''$ sexagesimal. Correcting the work of RHAETICUS and OTHO, BARTHOLOMAEUS PITISCUS (1561-1613) published at Frankfort in 1613 his *Thesaurus mathematicus sive canon sinuum* containing the natural values of sines and cosines for $r = 10^{15}$ with the same interval of $10''$ (2). But with the invention of logarithms the following year and the subsequent publishing of logarithm tables, the use of tables of natural values of trigonometric functions largely ceased until the recent invention of calculating machines made natural tables again desirable (3).

In the meantime modern analysis has supplied for the construction of such tables manifold methods of computing and checking results. Among the few tables besides those of RHAETICUS that have been computed *de novo* undoubtedly the most notable (4) are the *Nouvelles tables trigonométriques fondamentales* calculated « without personal or mechanical assistance » during the period 1908-1914 by H. ANDOYER of the University of Paris. We shall merely outline the method of computing the natural sines and cosines as given by the author (5).

(1) For further consideration of the work of PEURBACH and REGIOMONTANUS see p. 40-43. BRAUNMÜHL believes that the above method, evidently based on the Arabic, is due to the influence of AZ-ZARQALI (BRAUNMÜHL, p. 121).

(2) For detailed treatment of these works see BRAUNMÜHL, p. 212 and fol. RHAETICUS had himself begun the tables with $r = 10^{15}$. He was the first European to discard the arc for the angle by relating functions to the right triangle.

(3) Compare NAPIER *Memorial Volume*, p. 243 and fol. London, 1915.

(4) PETERS published in Berlin in 1911 a table of natural sines and cosines to 21 decimal places for every $10''$ sexagesimal.

(5) The entire work comprises about 1600 quarto pages. The logarithmic tables appeared in one volume in 1911. The natural tables are appearing in three volumes. Vol. 1 (1915) is described below; vol. 2 (1916) contains natural tangents and cotangents; vol. 3 will contain natural secants and cosecants and two auxiliary tables to assist in calculating cosecants and cotangents for angles less than 15° .

numerous methods that he used to check by other formulas the results already obtained.

Returning now to Hindu trigonometry, we find expressed in words relationships equivalent to $\sin^2 \alpha + \cos^2 \alpha = 1$ and $\sin^2 2\alpha + \text{versin}^2 2\alpha = (2 \sin \alpha)^2$ or $\sin \alpha = \sqrt{1/2(1 - \cos^2 \alpha)}$ where $r = 1$ (1). These formulas differ from PTOLEMY'S in the use of the *sinus* (half-chord) instead of the whole chord. It was by the use of this more powerful instrument that the Arabs were enabled to build on the Greek foundation of the auxiliary trigonometry of chords the complete and independent trigonometry of the sine.

About the beginning of the ninth century of our era the caliphs of Bagdad gathered together men of learning and set them to translating into Arabic the scientific works of the Hindus and the Greeks. Before the century closed the labors of AL-KHUWARIZMĪ (about 825), THABIT IBN QURRAH, and others had not only made these treasures accessible to the Arabs but had also given an impetus to independent investigation, as soon evidenced in the work of AL-BATTANĪ, ABU'L-WEFA, and later AT-TUSĪ.

In his *Opus astronomicum* (2) AL-BATTANĪ, known in medieval Latin works as ALBATEGNIUS (3), uses *sinus*, *sinus complementi* or *cosinus*, and *sinus versus* but follows closely the older Arabs in blending Greek and Hindu notions. He constructs his table of sines simply by halving the Ptolemaic chords. For determining the altitude of the sun by the shadow, *umbra extensa* (cotangent), of the *gnomon* (sun-dial) of 12 units length according to Hindu custom (cfr. VARĀHA MIHIRA, p. 34) he makes a table (4) for every 1° from 0° to 90° of parts and minutes

(1) VARĀHA MIHIRA, *Pañchasiddhāntikā*, edition THIBAUT, chap. IV, Verses 3, 4, p. 22.

(2) C. A. NALLINO, AL-BATTANĪ sive ALBATEGNI *opus astronomicum*, Pubblicazioni del Reale Osservatorio de Brera in Milano, N. XL, 1913, 1917. The text and annotations by NALLINO, an authority on Arabic science, are in Latin and occupy some 850 quarto pages.

(3) ABŪ ABD ALLAH MUḤAMMAD IBN JĀBIR AL-BATTĀNĪ, called the PTOLEMY of the Arabs, made observations in Ar-Raqqa 879-918 and died in 929.

(4) A similar table but with numerous discrepancies of 1' is found in the Latin translation by ATHELARD of Bath of AL-MADJIRIT'S revision of the tables of AL-KHUWARIZMĪ. (Compare NALLINO, Part II, p. 60, and SUTER, *Die Astronomischen Tafeln des MUḤAMMED IBN MŪSĀ AL-KWARIZMĪ*, p. 174. "Copenhagen, 1914.") That contained in the *Alfonsine Tables* shows inaccuracies amounting to 40'. (RICO Y SINOBAS, vol. IV, p. 8.)

(sexagesimal). The *umbra recta* (or *extensa*) and the *umbra versa* seem to have been viewed as things wholly apart from *sinus* and *cosinus*. Not until the xvth century were they placed by Europeans in the same category by the adoption of a common radius. In addition to the formulas of the *Almagest* AL-BATTANI has expressions equivalent to $c : a = \sin \gamma : \sin \alpha$ and $b : a = \sin \beta : \sin \alpha$ but that he knew the general sine law seems doubtful (1).

In his *Almagest* (2), ABU'L-WEFA (3) explains the character of his work thus : « Although this subject has been treated by HIPPARCHUS, APOLLONIUS, PTOLEMY, and other eminent men of the ancients, we have followed in this book a method indicated by none of them... We have avoided the known methods that are difficult to understand, such as those of the quadrilateral and the six quantities. With the object of arriving at results most easily and directly... we have opened new views and furnished new proofs. We have also added several propositions, very useful in astronomy, which they did not mention ».

We find the equivalents both in *sinus* and in chords of the formulas $2 \sin^2 \alpha/2 = 1 - \cos \alpha$ and $\sin 2\alpha = 2 \sin \alpha \cos \alpha$. He also obtains geometrically $\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta$ after having found by another method without being able to simplify

$$\sin(\alpha \pm \beta) = \sqrt{(\sin^2 \alpha - \sin^2 \alpha \sin^2 \beta)} \pm \sqrt{(\sin^2 \beta - \sin^2 \alpha \sin^2 \beta)}.$$

His proof of the last result is, however, very simple. That for $\alpha + \beta$ follows. Given arcs AB, BC (fig. 8), draw radii OA, OB, OC and to OA, OC the respective perpendiculars BT, BH and join HT. Then HT equals the *sinus* of arc AC. For prolong BT, BH to E, D and join ED. HT then equals half DE because T, H are the mid-points of BE, BD. Therefore arc DBE is twice arc CBA and HT is equal to the *sinus* of arc CA.

Furthermore the *umbra* is handled by him as a genuine trigonometric line and the following relations established : $\tan \alpha : 1 = \sin \alpha : \cos \alpha$; $\cot \alpha : 1 = \cos \alpha : \sin \alpha$; $\tan \alpha : \sec \alpha = \sin \alpha : 1$; $\tan \alpha : 1 = 1 : \cot \alpha$; $\sec \alpha = \sqrt{1 + \tan^2 \alpha}$; $\csc \alpha = \sqrt{1 + \cot^2 \alpha}$. Here we have replaced his *r* by unity. ABU'L-WEFA himself says :

(1) NALLINO, Part I, p. XLVII.

(2) CARRA DE VAUX, « L'Almageste d'ABU'L-WEFA ALBUZDJANI », *Journal Asiatique*, series VIII, vol. 10, 1892, p. 408-471.

(3) ABU'L-WEFA MUHAMMAD AL-BUZDJANI (940-998, Bagdad).

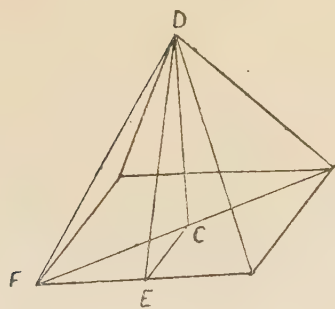


Fig. 1

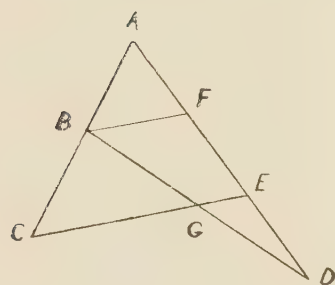


Fig. 3

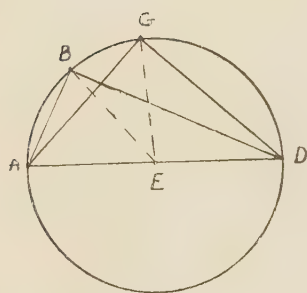


Fig. 5

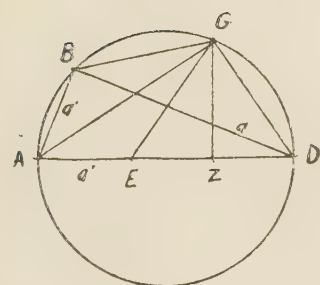


Fig. 7

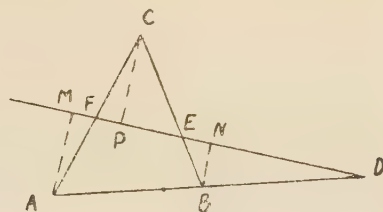


Fig. 2

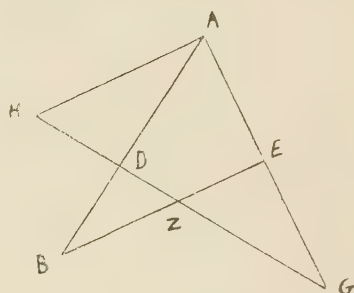


Fig. 4

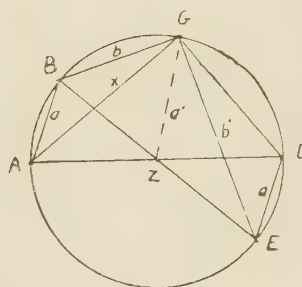


Fig. 6

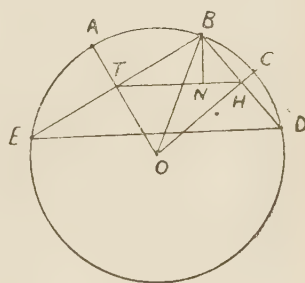


Fig. 8

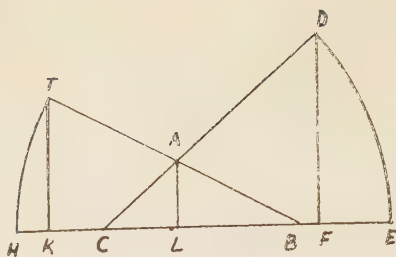


Fig. 9

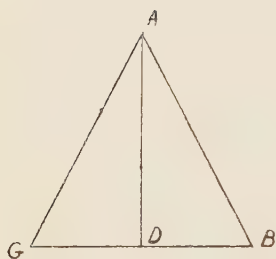


Fig. 11

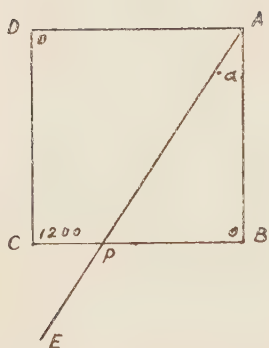


Fig. 13

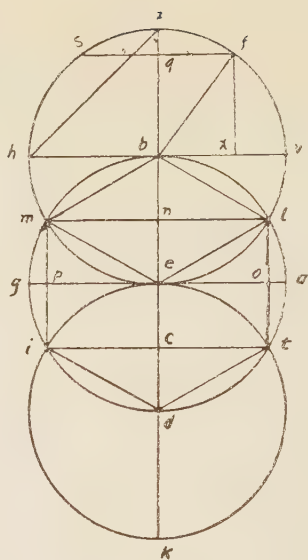


Fig. 10

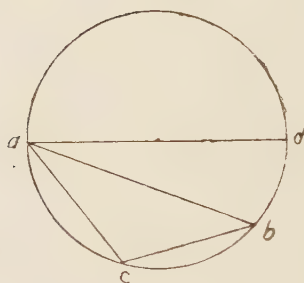


Fig. 12

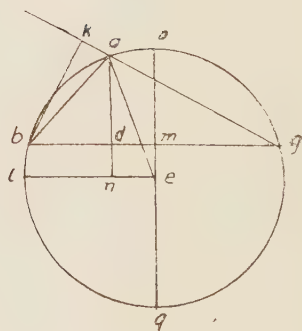


Fig. 14

« It is evident that if one takes the radius as unity the ratio of the sinus of an arc to the sinus of its complement is the tangent (first shadow), and the ratio of the sinus of the complement to the sinus of the arc is the cotangent (second shadow) ». (CARRA DE VAUX, p. 420.) Moreover he constructed a table of tangents and cotangents with the radius, subdivided sexagesimally, equal to unity, an idea ascribed by AT-TUSI to AL-BIRUNI (1). BRAUNMÜHL (p. 58) remarks that the above quotation can not be sufficiently emphasized, for in spite of its clear statement of the idea of setting $r = 1$, the radius was dragged along into the eighteenth century.

Contemporary with ABU'L-WEFA was IBN YUNUS (2), astronomer of Cairo and author of the *Hakimite Tables* which contained, according to HANKEL, the sum total of two centuries of development of Arabic astronomy and was the authoritative source for all later astronomers of the Orient. In his collection of rules, given without proof, is included the equivalent of the formula, $\cos \alpha \cos \beta = 1/2 [\cos(\alpha + \beta) + \cos(\alpha - \beta)]$ (3).

In addition to editing and revising many Greek and Arabic works, including a revision of the *Hakimite Tables*, AT-TUSI (4) wrote the important *Treatise on the Quadrilateral* (*Shakl al-Katta*). As previously mentioned, this is the first known work in which trigonometry is treated independently of astronomy. Moreover both plane and spherical trigonometry (5) are developed in great completeness.

The fourth book closes thus: « But the moderns, either from fear of confusion in the investigation of the different ratios and their varieties or to avoid the tedium involved in the use of composed ratios have devised easier methods to take the place of the quadrilateral. Also, once engaged in this study, we thought it useful to speak of these methods in order to give completeness to this branch of science ». Here for the first time in mathematical literature we find the explicit formulation of the sine law for the plane triangle. Two methods of proof for both the acute and the obtuse triangle are given. We shall reproduce one of the former.

(1) *Traité du Quadrilatère*, p. 165.

(2) ABŪ'L-HASAN ALIIBN YŪNUS (JŪNOS), active 1003.

(3) For probable method of proof see BRAUNMÜHL, p. 62-64.

(4) NĀSIR AD-DĪN AT-TŪSĪ (1201-1274, Bagdad).

(5) Of the 5 books of the *Treatise* the third is devoted to plane trigonometry and the fifth to spherical.

Let ABC (fig. 9) be the given triangle. Prolong CB to make $CE = 60 = R$. From the center C with radius CE describe an arc meeting CA produced at D , and from D drop DF perpendicular to CE . Similarly construct TK . Draw AL perpendicular to CB . Then by similar triangles $AB : AL = TB(R) : TK$ and $AL : AC = DF : DC(R)$. Thence $AB : AC = DF : TK = \sin ACB : \sin ABC$ (1).

Methods of solution of the triangle for all cases, with only one solution for the ambiguous case, are developed but where the sine law is inapplicable the procedure is to reduce to right triangles. An important feature of this work is the reproduction of the methods of proof of numerous predecessors of the author and statements concerning questions of priority. The chief difficulty encountered in studying the historical side of medieval mathematics is the failure of the authors to mention their sources of information.

We must now turn our attention to the West Arabs in Spain, for it was through this channel that mathematics flowed into medieval Europe. Although the East Arabs and the West Arabs possessed the same written language and religion, political animosities broke their free intercourse, so that the works of $ABU'L\text{-}WEFA$ and $AT\text{-}TUSI$ appear to have remained unknown in Europe to the time of $REGIOMONTANUS$. We shall find that science in Spain did not keep pace with the development in Bagdad.

$AZ\text{-}ZARQALI$ (2) was the most celebrated astronomer of his time. Under his direction a group of Moslem and Jewish scholars constructed the *Toledo Tables* which enjoyed an enormous circulation. The introduction to the tables, entitled *Canones sive regule super tabulas astronomie* (3) and containing besides the canons the explanation of the calculation of the tables, was written by $AZ\text{-}ZARQALI$ himself. $STEINSCHNEIDER$ records 48 Latin manuscripts of this work in European libraries besides numerous extracts and commentaries in Latin, Hebrew, and Arabic (4). He also observes that although the celebrated

(1) *Traité du Quadrilatère*, p. 70.

(2) $ABU\ I\dot{S}H\bar{A}Q\ AZ\text{-}ZARQ\bar{A}L\bar{I}$ ($ARZACHEL$ of the Latins) made astronomical observations in Toledo 1061-1080.

(3) *Bibliotheca Mathematica*, Third series, vol. 1, p. 337 and fol. Leipzig, 1900. CURTZE publishes here the Latin translation by GERARD of Cremona.

(4) *Bullettino di Bibliografia e di Storia delle Scienze Matematiche e Fisiche*, published by BONCOMPAGNI, vol. 20, p. 6-32. Rome, 1887. This is one of a series of articles by STEINSCHNEIDER on the life and work of $AZ\text{-}ZARQALI$.

Alfonsine Tables were constructed under the inspiration and guidance of King ALFONSO the Wise of Castile in the decade 1262-1272, most of these manuscripts were written after the middle of the XIIIth century. The climax of Az-ZARQALI's astronomical work was his *Safiha* (*saphea*) or universal astrolabe (1).

Having assumed the circumference of a circle to be 360° and the diameter 300', an innovation found in no previous work, Az-ZARQALI, following the Hindus, defines the *kardaga* (2) as the arc of 15° , the *sinus rectus* as the half-chord of twice the given arc, the *sinus totus* as the semi-diameter of 150', and the *sinus versus* as 150' less the *sinus* of the complement of the given arc (assumed $< 90^\circ$). He then proceeds to give substantially the following description of the accompanying figure (fig. 10). Let the lines *ga* and *bd* intersect orthogonally in the center *e* of the given circle *abgd*, and with the centers *b*, *d* and the given radius let two other circles be described intersecting *abgd* in the points *m*, *l*, *t*, *i*. Then chords *lb*, *bm*, *me*, *el* are all equal, as also their arcs, and *en*, *nb* are each half the radius. Similarly *dc* = *ce* = *en*. Hence *lt* = *mi* = *cn* = radius, and the circle *abgd* is divided into 6 equal parts.

Then *mp* (text gives *mg*) = sinus of $30^\circ = 75'$ and *mn* = sinus of 60° . Since *zh* = $\sqrt{45000}$ is the chord of 90° , sinus of $45^\circ = \sqrt{11250}$. To find the chord of arc *la* = 30° add the square of *lo*, the sinus of 30° to the square of *oa*, the difference between the radius and the sinus of 60° , and take the root (3) of the result *t*. Then *la/2* = sinus of 15° . Again if arc *fz* = arc *zs* = 15° , *fq* = sinus of 15° and *qb* = *fx* = sinus of 75° . Therefore sinus of $75^\circ = \sqrt{150^2 - fq^2}$. Now since 2 kardagas = 30° the sinus of 2 kardagas is $75'$ and the sinus of 4 kardagas is the root of the difference of the squares of the radius and the sinus of 2 kardagas. Similarly from *la/2*, the sinus of 1 kardaga, find the sinus of 5 kardagas. Then subtract the sinus of 15° from the sinus

(1) RICO Y SINOBAS, *Libros Alfonsies del Saber de Astronomia*, Madrid, 1864. The third of the five folio volumes is devoted to the construction and use of this instrument.

(2) AL-BIRUNI reports that *kardagia* used to mean an arc of $225'$, which is the arc the Hindus considered equal to its sine. The word is perhaps an alteration of the Sanscrit *cramad̐jya*, which signifies *sinus rectus*. (REINAUD, "Mémoire géog., hist. et scientifique sur l'Inde", *Mémoires, Académie des Inscriptions et Belles-Lettres*, vol. 18, p. 313.)

(3) Here and later the square root is to be understood.

of 30° and the residuum is the sinus of the second kardaga. The difference between the sinus of 30° and the sinus of 45° is the sinus of the third kardaga, and so on (1). We find also the equivalent of

$$\sin 7^\circ 30' = \sqrt{\sin 30^\circ (\sin 90^\circ - \sin 75^\circ)} = \sqrt{1/2 (1 - \cos 45^\circ)} \quad (2).$$

Then follow the rules for finding the sinus of $82^\circ 30'$, 45° , $22^\circ 30'$, $67^\circ 30'$, $11^\circ 15'$, $78^\circ 45'$, $37^\circ 30'$, $52^\circ 30'$.

Using the Hindu *gnomon* of 12 units length but retaining his own radius of $150'$, he finds the shadow of the sun from its altitude, and conversely, by formulas equivalent to

$$\cot \alpha = \frac{\cos \alpha}{\sin \alpha}, \quad \cos \alpha = \frac{\cot \alpha}{\sqrt{\cot^2 \alpha + 1}}, \quad \sin \alpha = \frac{1}{\sqrt{\cot^2 \alpha + 1}}.$$

As values of π he gives $22/7$, $\sqrt{10}$, $62832/2000$.

The work of JABIR (3), 9 books on astronomy, in the translation of GERARD OF CREMONA was published by PETER APIAN in 1534 (4). Proposition XXIV of the first book follows.

« And since this has already been explained let us begin to show how the unknown sides and angles of a triangle are found from the known, in order that we may avoid repetitions. Therefore I say that when in a rectilinear triangle *abg* (fig. 14) two sides *ab*, *bg* and the included angle *b* are known, the side *ag* and the two remaining angles are also known. And this is the proof. I shall draw from point *a* to line *bg* the perpendicular *ad*. Then because angle *abg* and line *bg* are known it is known whether *ad* will fall within or without *bg*. And since *d* is a right angle line *ab* is the diameter of a circle which contains triangle *abd*. Also since angle *b* is known the arc of this circle above *ad* will be known. Therefore the chord *ad* will be known from the measure (quantitatem) by which the diameter of the circle is known. Hence the line *bd* is known. Then since line *bd* is known

(1) This is doubtless closely connected with the Hindu rules for calculating their table of sines and the method of differences employed in these rules. See page 303 above.

(2) « Si autem voleris invenire sinum secundum minores circuli porcionis, sinum huius kardage sexte in sinum 30 graduum multiplica, et collecte inde summe quere radicem, que erit sinus 7 graduum et dimidii. » This, the complete explanation of the above equation, illustrates the absence of proofs in *Canones*.

(3) ABŪ MUḤAMMAD JĀBIR IBN AFLAḤ (GEBER of the Latins) was active in Seville between 1140 and 1150.

(4) *Instrumentum primi mobilis a PETRÒ APIANO... Accedunt iis Gebri filii Afla Hispalensis... libri IX de astronomia... Norimb. 1534, in folio.*

in relation to lines ab and bd line dg is also known. And since it has already been shown that line ad is known therefore must line ag be known. Therefore by the measure by which side ag is 120 is the perpendicular ad known. Therefore the arc ad in the circle containing triangle adg is known. Hence angle agd is known, and since angle abg was already known angle bag is known. Therefore all the sides and angles of the triangle are known and the proof is complete. »

We observe that JABIR divides the triangle into two right triangles about which he circumscribes circles and finds the unknown parts by the method of chords. The first sentence quoted above shows that he felt the need of an independent trigonometry but the chief value of the work concerns spherical trigonometry and astronomy and therefore need not detain us further.

Turning now to Christian Europe, we find hardly a trace of our science until the twelfth century, about the time Archbishop RAIMOND (between 1130 and 1150) instituted in Toledo a school of translators of Arabic works. From the time of Gerbert (940-1003, in 999 became Pope SYLVESTER II) who probably possessed Arabic works in Latin translation (1) interest in the philosophy of ARISTOTLE and consequently in astronomy and mathematics grew until, with the waning of Moslem power in Spain, the Christians, assisted by Jewish scholars, in the XIIIth century attained an activity in translating Arabic works that rivalled that of the Arabs themselves in their acquisition of the treasures of Greek and Hindu learning. The most prolific of these translators was GERARD OF CREMONA (1114-1187). Among the 76 works that he turned into Latin were PTOLEMY's *Almagest*, the *Canones* of AZ-ZARQALI, the *Spherics* of MENELAUS, and the works of JABIR and THABIT IBN QURRAH. JOHN OF SEVILLE (active 1135-1153), known also as JOHN OF LUNA and as JOHANNES HISPALENSIS, produced a Latin version of AL-FERGANI's (ALFRAGANUS in Latin) astronomy, PLATO of Tivoli (about 1136) translated AL-BATTANI's *De motu stellarum* and the *Spherics* of THEODOSIUS, and ATHELARD OF BATH (between 1120 and 1130), one of the earliest in this field of activity, made accessible to the Latins the astronomical tables of AL-KHUWARIZMI (2).

(1) DUHEM, *Le système du monde*, vol. 3, p. 163-165. Paris, 1915.

(2) See also HASKINS, "The Reception of Arabic Science in England", *The English Historical Review*, Jan. 1915, p. 56-69. Here are discussed the labors of WALCHER, PETRUS ANFUSI, ROBERT OF CHESTER, ROGER OF HEREFORD, DANIEL OF MORLEY, etc.

With the opening of the thirteenth century we observe the transition from mere translation to constructive mathematical production. In the year 1202 LEONARD OF PISA, called also FIBONACCI, published his *Liber Abaci*, of great importance in the history of arithmetic and algebra, and in 1220 his *Practica Geometriæ* (1), in which for the first time surveying by trigonometric methods is reduced to a mathematical basis. While earlier works on surveying had been for the most part a collection of rules, LEONARD OF PISA gave derivations and proofs. His knowledge of trigonometry is derived from PROLEMY but he defines the *sinus rectus* and *sagitta* (sinus versus) like the Arabs. His table of chords and arcs is constructed with a diameter of 42 *perticæ* and circumference 132 *perticæ*, where 1 *pertica* (measuring rod) = 6 *pedum* = 108 *unciarum* = 2160 *punctorum* (2). With his quadrant (3), constructed after the manner of the Arabs, can be read from the position of the plumb-line the *umbra recta* and the *umbra versa* though the term *umbra* does not appear (4). However, in the work of ROBERT THE ENGLISHMAN (about 1231, Montpellier) we find expressed between the two *umbræ* the relation which is equivalent to $\tan \alpha \cot \alpha = 1$.

Apparently the *Toledo Tables* were little known to the Latins until they appeared in the writings of WILLIAM THE ENGLISHMAN (about 1234, Marseilles) (5). We have already referred (p. 313) to the important astronomical compilation of which the *Alfonsine Tables* formed a part. We may add here that LUCAS GAURICUS (1476-1558), an Italian astrologer and a prolific writer, attributes to CAMPANUS OF NOVARA (active about 1260-1280) a true table of tangents for each degree from 0° to 45°. This antedates by about two centuries the *tabula fœconda* (see p. 322) of REGIOMONTANUS which, apparently with good reason notwithstanding GAURICUS' statement, has been supposed the earliest constructed in Europe.

An anonymous manuscript bearing close resemblance to the *Quadrupartitum* was published by CURTZE, who shows that the supposed

(1) BONCOMPAGNI, *Scritti di LEONARDO PISANO del Secolo Decimoterzo*. Rome, 1862. *Practica Geometriæ* is the second volume of the *Scritti*.

(2) *Practica Geometriæ*, p. 95.

(3) Same, p. 206.

(4) Compare BRAUNMÜHL, p. 96-98.

(5) DUHEM, vol. 3, p. 315. In the Latin texts WILLIAM is often designated simply as "MARSILIENSIS".

author was only a commentator and that instead of belonging to the fifteenth century as was formerly supposed the manuscript belongs to the thirteenth century (1). It occupies folios 83^a — 103^b of an excellent manuscript of 127 leaves containing 11 other works. We find in it the terms *gardaga* (kardaga, p. 313 above), *cada* (catha, p. 299) (2), *sinus rectus*, *sinus versus*. The diameter is assumed to be 300 minutes (300 minutorum) for sines and 120 degrees (120 graduum) for chords. There is one numerical and one linear demonstration of a ratio composed of ratios. The figure illustrating the application to the *catha* and labelled *cada coniuncta* corresponds to WALLINGFORD'S *catha disiuncta* and there is no figure given in the second case. The application of PROLEMY'S Theorem to the case when one side of the quadrilateral is a diameter of the circle and to finding the chord of the sum and of the difference of the arcs of two given chords concludes the part preceding the astronomical section, that is to folio 87^a. The latter part of this treatise contains a section on the *umbra*, a subject not found in the *Quadripartitum*. Now WALLINGFORD devotes two-thirds of the 46 pages of his work to laying the mathematical foundation whereas the other uses only 8 and in these 8 pages no author or theorem is cited. Our conclusion is that the two works can be only indirectly connected through common sources; a single source seems quite improbable.

(1) *Bibliotheca Mathematica*, Third Series, vol. 1, Leipzig, 1900. Under the title : "Urkunde zur Geschichte der Trigonometrie im christlicher Mittel alter", CURTZE gives the Latin texts of parts of 8 important manuscripts as follows, the fourth being that under consideration above :

1. Aus dem *liber embadorum* des SAVASORDA in der Uebersetzung des PLATO VON TIVOLI, p. 321-337.
2. Aus dem *Scripta MARSILIENSIS super Canones ARZACHELIS*, p. 347-353. (Compare page 312 above.)
3. Aus den *Canones sive regule super tabulas Toletanas* des AL-ZARKÂLÎ, p. 337-347.
4. *Anonyme Abhandlung über Trigonometrie aus dem Ende des XIII Jahrhunderts*, p. 353-372.
5. Aus LEO DE BALNEOLIS ISRAHELITA *de sinibus, chordis et arcubus, item instrumento revelatore secretorum* p. 372-380.
6. *Anonyme Abhandlung De tribus notis*, p. 380-390.
7. Die *Canones Tabularum primi mobilis* des JOHANNES DE LINERIIS, p. 390-413.
8. Die *Sinusrechnung* des JOHANNES DE MURIS, p. 413-416.

(2) Both CURTZE and BRAUNMÜHL say that the variation of the consonants indicates Saxon authorship.

To the year 1321 belongs LEO DE BALNEOLIS ISRAHELITA *de sinibus, chordis et arcubus...*, (1) of LEVI BEN GERSON (died 1344, Avignon), known also as LEO DE BALNEOLIS. It contains 9 chapters, of which the second, composed of 5 parts (*dictiones*), is of special interest for the history of trigonometry. The *dictio prima* gives definitions and explanations of degrees, minutes, etc., signs of the zodiac, division of the diameter into 120 degrees (« not equal to the degrees of the circumference »), arc, chord, *sinus*, and *sagitta*. The *dictio secunda* contains the derivation of the relations which we write thus :

$$\begin{aligned}\text{versin } \alpha + \cos \alpha &= 1, \quad \text{versin}(90^\circ + \alpha) = 1 + \sin \alpha, \\ \text{chd}^2(\alpha \pm \beta) &= (\sin \alpha \pm \sin \beta)^2 + (\text{versin } \alpha - \text{versin } \beta)^2;\end{aligned}$$

the relation $\text{chd}^2 \alpha = \text{sagitta} \cdot \text{diameter}$; and the fact that the chord of the half arc is known from the chord of the arc. From these relations LEVI BEN GERSON constructs a table of sines for intervals of 15'. From the sines of 90°, 60°, 36° are obtained besides the sines of the halves of these angles the sines of 24°, 6°, 8°15' (6° + 2°15'), and finally of 15' + 1/128° (33/128°) and 15' - 1/64° (15/64°). Thence is found the *sinus* of 15' = 0°15'42''28'''12^{iv}27^v, a value correct to 7 decimal places. Some further calculation concludes *dictio tertia*. In *dictio quarta* is given the *Tabula sinus* and the explanation of its use for finding the *sinus* and the *sagitta* from the arc, and conversely. *Dictio quinta* (2), containing the theory of the solution of triangles, is an important independent contribution by LEVI BEN GERSON. It contains the following propositions :

(I) Given two lines in a right triangle, the remaining parts are known.

(II) Given the three sides of any triangle, all the angles are known.

(III) Given two sides and the angle opposite one of them, the remaining side and angles are known. (Only one solution.)

Corollary A. « In every triangle one side has to another the same ratio that one sinus of the angles subtended by the lines has to the other ».

Corollary B. If the angles and one side of a triangle are known, so are the remaining sides.

(1) See note, p. 317. This work was translated from Hebrew into Latin in 1342, a fact to which we shall later have occasion to refer.

(2) This part is not given in the above citation but was published by CURTZE in *Bibliotheca Mathematica*, 1898, 2nd series, vol. 12, p. 103-106.

(IV) Given two sides and the included angle of a triangle, the remaining parts are known.

Here appears for the first time in the mathematical literature of Europe the clear and definite statement of the sine law. Also the four theorems here presented show great contrast with a similar work on the solution of triangles, *De tribus notis* (1), written about the same year and containing separate treatment of 25 different cases. The method of *De tribus notis* is that of JABIR, that is to divide each triangle into two right triangles and solve by means of the chords and arcs of the circumscribed circles. LEVI BEN GERSON uses the Pythagorean Theorem and the *sinus* in the first case; in the second he determines the height by the Euclidean method and then applies the *sinus*. We give below a rather free translation of his proof of the third case. He uses the same figure in the fourth (2).

In triangle abc (fig. 12) let the lines ab , bc and the angle bac be known. I say that the line ac and the remaining angles are known. To prove this let a circle be passed through the vertices of abc and let ad be the diameter of this circle. Because angle bac is known and inscribed in the circle of which 360 degrees subtends two right angles therefore bc will be known and from the tables of arcs and sines we shall know the chord bc in terms of the 120 degrees of ad . So we know the ratio of cb to ad . Then from the known ratio of bc to ab is known the ratio of ab to ad . Then we know the value of ab in degrees (diametral) and from the tables we shall thus know arc abc . Therefore we know arc ac , the difference of the known arcs. Hence from the tables we know angle abc . Finally from the two known angles we know the third angle. Therefore etc.

The *Canones tabularum primi mobilis* (3) of JOHANNES DE LINERHS, without mentioning any source, employs verbatim the method of calculating the *sinus* found in the *Canones* of AZ-ZARQALI. Among the tables are *tabule sinuum et cordarum*, *tabula umbre*, and *tabula proportionis*. JOHN OF MEURS, in the last of the CURTZE group of manuscripts, shows some independence by constructing a figure of his own to replace that handed down from AZ-ZARQALI. The *Practica geometriæ* shows that its author DOMINICUS DE CLAVASIO (active 1349-1357, Paris) was acquainted with *sinus*, *sinus versus*, *complementum*

(1) See note page 317.

(2) Compare BRAUNMÜHL, p. 103-106.

(3) Written in Paris, 1322. See note, p. 317.

sinus versus, *umbra recta*, *umbra versa* and well versed in EUCLID, PTOLEMY and WITelo. Moreover he gave strict proofs for the rules he set up for surveying (1).

JOHANN SCHINDEL (1380?-1442), known also as JOHANNES DE GEMUNDEN, wrote JOHANNES DE GEMUNDEN *de sinibus, chordis et arcubus* in which appear the terms *sinus rectus*, *sinus versus* and *sagitta*, *sinus totus* or *perfectus*, *cardaga*. As in our tables he uses proportional parts and gives the differences. With him originates the radius of 600 000 though the idea of combining decimal and sexagesimal measurement may have been due to JOHANNES DE LINERHIS. It remained for GEORG PEURBACH (1423-1461, Vienna) to construct a complete table of sines for intervals of 10' with the above radius (2). It is interesting to note that PEURBACH was the first to employ the « geometric square » for astronomical as well as for terrestrial measurements (3). Instead of the usual 12 units for the length of BC (fig. 13) he uses 1200. He finds

$$\sin \alpha = \left[\frac{a}{\sqrt{a^2 + 1200^2}} \right] \cdot 600000$$

where $a = BP$. So we really have a table of arctangents, since $\alpha = \arctan a/1200$.

JOHANNES MÜLLER (1436-1476), generally known as REGIOMONTANUS from the Latin name of his birth place, Königsberg, became at the age of 15 years a pupil of PEURBACH. In intimate association and collaboration with his teacher during the last ten years of PEURBACH's life REGIOMONTANUS gained the necessary inspiration and equipment for the great labors that lay before him. In his extensive travels he made the acquaintance of the Italian astronomers as well as those of his own country. He was well versed in Greek and Latin and as early as 1464 he lectured on the Arabic astronomer AL-FERGANI. Indeed it was from Arabic sources that he drew most of his

(1) CURTZE, « Ueber den DOMINICUS PARISIENSIS der *Geometria Culmensis* », *Bibliotheca Mathematica*, 1895, p. 107-110. CURTZE says: « The *Practica Geometria* reveals DOMINICUS as an able mathematician who for his time developed noteworthy accomplishments ».

(2) This table was not published because the more accurate table of REGIOMONTANUS appeared before PEURBACH's *Tractatus* went to press. (BRAUNMÜHL, p. 116.)

(3) CURTZE, « Ueber die Mittelalter zur Feldmessung benutzten Instrumenten », *Bibliotheca Math.*, 1896, p. 65-72.

knowledge of trigonometry (1). His short but very active career came to an end in 1476 while he was in Rome to reform the calendar for Pope SIXTUS IV.

We have already mentioned his construction of trigonometric tables (p. 304). To his *Tabulæ directionum* (2) is appended a table of sines to radius 60 000 for intervals of one minute. This we may consider as the forerunner of the popular 5-place tables of the nineteenth century. With the *Tractatus* GEORGII PUREBACHII *super propositiones* PTOLEMÆI *de sinibus et chordis* which contains the familiar figure of AZ-ZARQALI and 6 propositions for finding the chord of any arc is commonly bound REGIOMONTANUS' *Compositio tabularum sinuum*. This work also contains 6 propositions: (I) Given the sinus of an arc, to find the sinus of its complement. (II) To find the sinus of arcs by kardagas. (III) The sinus of an arc ($< 90^\circ$) is the mean proportional between $r/2$ and the sinus versus of twice the arc. (IV) To find the side of regular pentagon and decagon. This is followed by a table, for $r = 600\,000\,000$, of the 96 values of the sine that can be readily obtained by applying the preceding propositions. (V) To find the side of the quindecagon. (VI) Ordinates of equal arcs are farther apart toward the center of the circle. Then comes the table of sines for $r = 6\,000\,000$ followed by the table for $r = 10\,000\,000$, the interval in each case being one minute.

The first systematic treatise on trigonometry published in Europe is JOHANNES DE MONTEREGIO *de triangulis omnimodis libri quinque* (3). Of the five books the first two deal with plane and the last three with spherical triangles. Of the 57 propositions of book I the first 19 treat of proportion and the rest deal geometrically with the finding of unknown parts in right, isosceles, and scalene triangles. Following each proposition is an explanation of its use and often a numerical illustration is also given. Only in proposition 20 is there mention of a trigonometric function: « In any right triangle, if you describe from one acute angle a circle with the greatest side as radius the side

(1) BRAUNMÜHL concludes after careful examination of the matter that his sources were the works of AL-BATTANI, AL-FERGANI, AZ-ZARQALI and JABIR. (« NASSIR EDDIN TUSI und REGIOMONTAN », p. 38).

(2) Written about 1467 and published in 1490 by RATDOLT in Augsburg.

(3) Written about 1464 and published in Nuremberg in 1533. The *Compositio* was published there in 1541. The three works are in the University of Michigan library.

opposite this angle will be the sinus rectus of its arc; but the third side will be the sinus rectus of the complementary arc.» The first of the 33 propositions of book II is the sine law. The one figure and the proof resemble Ar-Tusi's given above (p. 312) but it has not been proved that the *Treatise on the Quadrilateral* was known to REGIOMONTANUS. The statement and proof of proposition 25 follow.

« If in any triangle we know the base, the opposite angle and the perpendicular (altitude) or the area we shall at once know each of the sides. In the triangle abg (fig. 14) let the base bg , the perpendicular ad , or the area since the one depends on the other, and the angle bag be known. I say that we can know each of the sides. For let angle bag be obtuse and let the perpendicular from b meet ga produced at k . Then since angle bak is known the ratio of ab to bk is known by (blank) of this (book). But this ratio equals the ratio of ab times ag to bk times ag . Then since bk times ag is twice the area of triangle abg it is known and therefore the rectangle (product) of ab and ag is known. Now by the preceding proposition the diameter of the circumscribed circle is known from the perpendicular ad and the rectangle of ab and ag . Let the diameter pq meet the chord bg in m and let the radii eg , ea be drawn, as also el parallel to bg and meeting ad produced in n . Then dn equals em which is known from eg , mg and right angle emg . Hence an is known. Then ne is known from an , ne and right angle ane . Thus is known its equal dm and consequently the smaller segment bd which is the difference between bm and dm and the larger segment dq which is their sum are known. From these two segments and the perpendicular ad will be known the two sides » (1).

It is evident that REGIOMONTANUS was interested in a unified basis and systematic arrangement of trigonometric theory with the object of making it useful to the astronomer. That he early saw the advantage of the tangent function is evidenced by the appearance in his *Tabulae directionum* of a table of tangents for every degree from 0° to 90° with $r = 100\,000$. He called it the *tabula faconda*, that is

(1) We can find ab and ag from the two equations $bg^2 = ab^2 + ag^2 - 2 \cdot ab \cdot ag \cdot \cos bag$, $ab \cdot ag \cdot \sin bag = ad \cdot bg$. The first implicit statement of the formula for the area of a triangle in terms of two sides and the sine of the included angle appears to be proposition 26: « The area of a triangle and the product of two sides being known, either the included angle or its supplement will emerge ».

the *fruitful* (and therefore very useful) *table*, and the name was afterwards generally adopted to distinguish this table from the *tabulæ umbrarum*. The influence of REGIOMONTANUS' work on the progress of science was very great; his *De triangulis* ushered in a new era in the development of trigonometry.

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ISIS

International Review devoted to the History
:: of Science and Civilization ::

EDITED BY

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N° 11, Vol. IV (2) 1922

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